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# On dissipationless shock waves in a discrete nonlinear Schrödinger equation

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## Abstract

It is shown that the generalized discrete nonlinear Schrödinger equation in a small amplitude approximation is reduced to a number of basic nonlinear integrable equations, such as the KdV, mKdV and KdV(2) equations, or to the fifth-order KdV equation, depending on the values of the parameters. In the dispersionless limit these equations lead to the wave-breaking phenomenon for general enough initial conditions, and, after taking into account small dispersion effects, result in the formation of dissipationless shock waves. The Whitham theory of modulations of nonlinear waves is used for an analytical description of such waves. Numerical simulations are used to obtain different types of bright and dark shocks.

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## 1. Introduction

Dissipationless shock waves have been experimentally observed or their existence has been theoretically predicted in various nonlinear media such as water [1], plasma [2], optical fibres [3] and lattices [4]. In contrast to usual dissipative shocks where the combined action of nonlinear and dissipation effects leads to sharp jumps of the wave intensity, accompanied by abrupt changes of other wave characteristics, in dissipationless shocks, the viscosity effect is negligibly small compared to the dispersive one, and, instead of intensity jumps, the combined action of nonlinear and dispersion effects leads to the formation of an oscillatory wave region (for a review see, e.g., [5]). Since the intrinsic discreteness of a solid state system gives origin to strong dispersion, which can dominate dissipative effects in wave phenomena, it is of considerable interest to investigate details of the formation and dynamics of dissipationless shocks in lattices.

As a model, in the present paper we choose the generalized discrete nonlinear Schrödinger (GDNLS) equation

$$i\dot{q}_n + (1 - \eta|q_n|^2)(q_{n+1} + q_{n-1} - 2q_n) + 2(\rho^2 - |q_n|^2)q_n = 0 \quad (1)$$

introduced by Salerno [7, 8]. This equation represents a generalization of the simple tight-binding linear Schrödinger model for the dynamics of quasiparticles in a molecular crystal. Also it appears as a model for an electrical lattice [9]. An important mathematical property of equation (1) is that it provides a one-parametric transition between an integrable Ablowitz–Ladik (AL) model and the so-called discrete nonlinear Schrödinger (DNLS) equation ( $\eta = 1$  and  $\eta = 0$ , respectively).

In the present paper, we show that equation (1) can be linked in the small amplitude limit, not only to the Korteweg–de Vries (KdV) equation (the case explored in [10, 11]) but also to other integrable models: the modified KdV (mKdV) and KdV(2) equations as well as the fifth-order KdV equation (the last one not being integrable). It is worth emphasizing here that to the best of the authors' knowledge this is the first example of a 'physical' significance of the KdV(2) model, considered so far as a purely mathematical object interesting from the point of view of integrable systems.

Obviously, equation (1) has a constant amplitude solution  $q_n = \rho$ . In a small amplitude,  $|a| \ll \rho$ , and a long wave, when the discrete site index  $n$  can be replaced by a continuous coordinate  $x$  limit, the evolution of small amplitude perturbations against this constant background, i.e. of excitations of the type

$$x = nh \quad q_n(t) = q(x, t) \quad (2)$$

$$q(x, t) = (\rho + a(x, t)) \exp(-i\phi(x, t)) \quad (3)$$

where we will assume that the lattice parameter  $h = 1$ , is governed by the KdV equation for the amplitude  $a(x, t)$  (see [10, 11]):

$$a_t - \frac{2(3 - 4\eta\rho^2)}{\sqrt{1 - \eta\rho^2}} a a_x + \frac{\sqrt{1 - \eta\rho^2}}{12\rho} [3(1 - \eta\rho^2) - \rho^2] a_{xxx} = 0 \quad (4)$$

which is written in the reference system moving with velocity  $2\rho\sqrt{1 - \eta\rho^2}$  of linear waves in the dispersionless limit. It is well known (see, e.g., [1]) that if the initial pulse is smooth enough, so that the nonlinear term dominates the dispersive one at the initial stage of the evolution, then the dissipationless shock wave develops. The theory of such waves, described by the KdV equation, is well developed (see, e.g., [5]). The existence of the respective shock waves for model (1) has been predicted analytically and observed in numerical simulations in [10, 11]. Moreover, as shown in [12] a discrete nonlinear Schrödinger equation of a rather general type can bear 'KdV-type' shock waves. In this context the results presented in the present paper although being mostly related to model (1) display some general characteristic features of lattices of the nonlinear Schrödinger type.

The coefficients of equation (4), namely

$$C_{\text{NL}} = -\frac{2(3 - 4\eta\rho^2)}{\sqrt{1 - \eta\rho^2}} \quad (5)$$

and

$$C_{\text{DI}} = \frac{\sqrt{1 - \eta\rho^2}}{12\rho} [3(1 - \eta\rho^2) - \rho^2] \quad (6)$$

below referred to as nonlinearity and dispersion respectively, depend on two parameters  $\eta$  and  $\rho$  and can vanish at a special choice of these parameters. Then one has to consider corrections

to the KdV equation. This is the situation studied in the present paper. More specifically, in section 2 we derive evolution equations at the small-amplitude limit for the whole region of parameters  $\eta$  and  $\rho$  and show that for  $C_{NL} = 0$  the GDNLS equation reduces to the mKdV equation (section 2.2), for  $C_{DI} = 0$  to a nonlinear equation with dispersion of the fifth order (section 2.3), and at the point where both  $C_{NL} = 0$  and  $C_{DI} = 0$  to the KdV(2) equation (section 2.4). As we mentioned above, the last result sheds some new light on the nature of higher equations of the KdV hierarchy—they arise as small amplitude approximations to completely integrable equations, if lower orders of nonlinear and dispersion contributions vanish at some values of the parameters of the equation under consideration.

The next aim of the paper—developing a theory of shock waves—is realized in section 3 for the mKdV (section 3.1) and KdV(2) (section 3.2) equations, analogous to those developed earlier for the KdV equation, where we discuss the possibility of the existence of shock waves. The theory developed in this section permits one to describe in detail the behaviour of shocks after the wave-breaking point for different values of the parameters entering equation (1).

Finally in section 4, we perform a numerical analysis of the GDNLS equation showing different behaviour of the dynamics of shocks depending on the value of parameter  $\rho$ . The goal of the numerical simulations is to show that changing parameter  $\rho$  along a line  $\eta = \text{const}$ , one can obtain different types of shock waves corroborating the description obtained from the small-amplitude multiscale expansion.

The results are summarized in section 5.

## 2. Small amplitude approximation

Using ansatz (3), we rewrite equation (1) as

$$ia_t + (\rho + a)\phi_t - 4\rho^2 a - 6\rho a^2 + [1 - \eta(\rho + a)^2] \times \{(\rho + a(x + 1, t)) \exp[-i(\phi(x + 1, t) - \phi(x, t))] + (\rho + a(x - 1, t)) \exp[-i(\phi(x - 1, t) - \phi(x, t))]\} = 0 \tag{7}$$

i.e. where  $|a| \ll \rho$  and  $|\phi(x + 1, t) - \phi(x, t)| \ll 1$ . In the linear approximation, this equation yields for the harmonic wave solution

$$a(x, t) \propto \exp[i(Kx - \Omega t)] \quad \phi(x, t) \propto \exp[i(Kx - \Omega t)]$$

the dispersion relation [10, 11]

$$\Omega = \pm 4\sqrt{1 - \eta\rho^2} \sin \frac{K}{2} \left[ \rho^2 + (1 - \eta\rho^2) \sin^2 \frac{K}{2} \right]^{1/2} \tag{8}$$

$$\cong \pm 2\rho\sqrt{1 - \eta\rho^2} K \left[ 1 + \frac{3(1 - \eta\rho^2) - \rho^2}{24\rho^2} K^2 + O(K^4) \right] \tag{9}$$

where the expansion in powers of  $K$  corresponds to taking into account different orders of the dispersion effects. In the lowest order, when the dispersion effects are neglected, linear waves propagate with a constant velocity

$$v = \pm 2\rho\sqrt{1 - \eta\rho^2}. \tag{10}$$

To evaluate the contribution of small (for  $|a| \ll \rho$ ) nonlinear effects, it is convenient to introduce a small parameter  $\varepsilon \sim a$  and pass to such scaled variables in which nonlinear and dispersion effects make contributions of the same order of magnitude to the evolution of the wave. Since the choice of these scaled variables depends on the values of the parameters  $\rho$  and  $\eta$ , we consider the different relevant cases separately.

### 2.1. KdV equation

For the sake of completeness we start by reproducing briefly some results of [11]. We expand  $a(x \pm 1, t)$  and  $\phi(x \pm 1, t)$  into Taylor series around  $x$ , introduce scaling indices  $\{\alpha, \beta, \gamma\}$

$$a \sim \varepsilon \quad t \sim \varepsilon^{-\alpha} \quad x \sim \varepsilon^{-\beta} \quad \phi \sim \varepsilon^\gamma \quad (11)$$

and demand that in the reference frame moving with velocity (10) of linear waves the lowest quadratic nonlinearity has the same order of magnitude as the second term in expansion (8) of the dispersion relation,  $a \sim \phi_x$ ,  $a_t \sim aa_x \sim a_{xxx}$ , which yields  $\{\alpha = \frac{3}{2}, \beta = \gamma = \frac{1}{2}\}$ . Thus, the scaled variables have the form

$$\tau = \varepsilon^{3/2}t \quad \xi = \varepsilon^{1/2}(x + vt) \quad v = \pm 2\rho\sqrt{1 - \eta\rho^2} \quad (12)$$

and  $a(x, t)$  and  $\phi(x, t)$  should be looked for in the form of expansions

$$\begin{aligned} a &= \varepsilon a^{(1)} + \varepsilon^2 a^{(2)} + \varepsilon^3 a^{(3)} + \dots \\ \phi &= \varepsilon^{1/2} \phi^{(1)} + \varepsilon^{3/2} \phi^{(2)} + \varepsilon^{5/2} \phi^{(2)} + \dots \end{aligned} \quad (13)$$

Then in the lowest order in the expansion of equation (7) in powers of  $\varepsilon$  we obtain the relationship

$$\phi_\xi^{(1)} = \frac{4\rho}{v} a^{(1)} \quad (14)$$

where we have chosen the upper sign of  $v$  in (12). Note that this first order gives us a relation between the two unknown variables  $a^{(1)}$  and  $\phi^{(1)}$  which will be useful to obtain at second order an equation for the only variable  $a^{(1)}$ . In the next order one arrives at the KdV equation (4) written in terms of the scaled variables, i.e. with  $a, t$  and  $x$  substituted by  $a^{(1)}, \tau$  and  $\xi$ , respectively:

$$a_t^{(1)} - \frac{2(3 - 4\eta\rho^2)}{\sqrt{1 - \eta\rho^2}} a^{(1)} a_x^{(1)} + \frac{\sqrt{1 - \eta\rho^2}}{12\rho} [3(1 - \eta\rho^2) - \rho^2] a_{xxx}^{(1)} = 0. \quad (15)$$

The nonlinearity (5) changes sign for

$$\eta = \frac{3}{4\rho^2} \quad (16)$$

and the dispersion term changes sign for

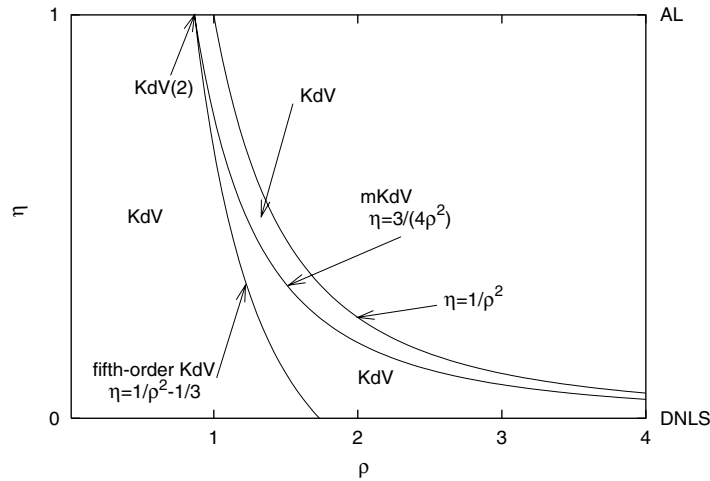
$$\eta = \frac{1}{\rho^2} - \frac{1}{3}. \quad (17)$$

Hence, there can be either bright solitons against a background ( $a(x, t) > 0$ ) or dark solitons ( $a(x, t) < 0$ ) of the GDNLS equation approximated by the KdV equation (4) (see figure 1). In both cases it is possible to pass to a new dependent variable  $u$  defined as

$$a^{(1)} = -\frac{1 - \eta\rho^2}{12\rho - 16\eta\rho^3} [3(1 - \eta\rho^2) - \rho^2] u \quad (18)$$

and change time as  $t \rightarrow \frac{\sqrt{1 - \eta\rho^2}}{12\rho} [3(1 - \eta\rho^2) - \rho^2] t$  such that the KdV equation takes its standard form

$$u_t + 6uu_x + u_{xxx} = 0. \quad (19)$$



**Figure 1.** Diagram showing different continuous limits of the GDNLS equation. The abbreviations are explained in the text.

2.2. *mKdV equation*

Exactly on the line (16) the nonlinearity vanishes:  $C_{NL} = 0$ . This means that the dispersion can no longer be balanced by the quadratic nonlinearity. Hence, one may expect that the modified KdV (mKdV) equation with cubic nonlinearity arises for values of  $\rho$  and  $\eta$  related by (16).

In order to check whether those qualitative predictions are correct one has to change the scaling. The new scaling should be chosen to take into account that the cubic nonlinearity now must have the same order of magnitude as the dispersion  $a_{xxx}$ , i.e.  $a \sim \phi_x, a_t \sim a^2 a_x \sim a_{xxx}$ . Then, using scaling (11) we find  $\{\alpha = 3, \beta = 1, \gamma = 0\}$ , so that instead of (12) we have the following scaled variables,

$$\tau = \varepsilon^3 t \quad \xi = \varepsilon(x + vt) \quad v = \pm \rho \tag{20}$$

where the value of the velocity is found by substituting (16) into (10). Now the solution is searched for in a form of the expansion

$$a = \varepsilon a^{(1)} + \varepsilon^2 a^{(2)} + \varepsilon^3 a^{(3)} + \dots \tag{21}$$

$$\phi = \phi^{(1)} + \varepsilon \phi^{(2)} + \varepsilon^2 \phi^{(3)} + \dots \tag{22}$$

In the lowest order of  $\varepsilon$  we obtain

$$\phi_\xi^{(1)} = 4a^{(1)} \tag{23}$$

which coincides with equation (14) after the substitution of  $v = \rho$ . In the next order we get the relationship

$$\phi_\xi^{(2)} = 4a^{(2)} + \frac{6}{\rho} a^{(1)2} \tag{24}$$

and finally in the highest relevant order we obtain the mKdV equation

$$a_\tau^{(1)} + \left(4\rho + \frac{21}{\rho}\right) a^{(1)2} a_\xi^{(1)} + \frac{1}{32} \left(\frac{1}{\rho} - \frac{4\rho}{3}\right) a_{\xi\xi\xi}^{(1)} = 0. \tag{25}$$

Since along the line (16) we have  $\rho > \sqrt{3}/2$ , the coefficient before the dispersion term is

always negative and hence equation (25) can be transformed into the following standard form,

$$u_t - 6u^2u_x + u_{xxx} = 0 \quad (26)$$

where  $u$  is given by

$$a^{(1)} = \frac{1}{4} \sqrt{\frac{4\rho^2 - 3}{4\rho^2 + 21}} u \quad (27)$$

and the time variable is renormalized

$$t \rightarrow \frac{1}{32\rho} \left(1 - \frac{4}{3}\rho^2\right) t \quad (28)$$

i.e. time in (26) is negative.

### 2.3. Fifth-order KdV equation

Along the line (17), the dispersion term in (4) vanishes,  $C_{DI} = 0$ . This means that if initial scaling of the wave packet is chosen according to (12), (13), the balance between the nonlinearity and dispersion is broken again, the nonlinearity becomes dominating. Such a pulse displays steepening until the higher order dispersion starts to balance the nonlinearity. In this case one expects the evolution equation for  $a(x, t)$  to contain quadratic nonlinear terms and the linear dispersion term is the one with the fifth-order space derivative of  $a(x, t)$ . Since the first-order dispersion effects in equation (4) disappear, the scaling should be chosen so that the quadratic nonlinearity has the same order of magnitude as  $a^{(V)}$ ,  $a \sim \phi_x$ ,  $a_t \sim aa_x \sim a^{(V)}$ , which yields  $\{\alpha = \frac{5}{4}, \beta = \frac{1}{4}, \gamma = \frac{3}{4}\}$ , so that the scaled variables are now given by

$$\tau = \varepsilon^{5/4} t \quad \xi = \varepsilon^{1/4} (x + vt) \quad v = \pm \frac{2\rho^2}{\sqrt{3}} \quad (29)$$

where velocity  $v$  is found by substituting (17) into (10). Now the condition of cancellation of terms in the second order demands that expansions of  $a$  and  $\phi$  have the form

$$\begin{aligned} a &= \varepsilon a^{(1)} + \varepsilon^{3/2} a^{(2)} + \varepsilon^2 a^{(3)} + \dots \\ \phi &= \varepsilon^{3/4} \phi^{(1)} + \varepsilon^{5/4} \phi^{(2)} + \varepsilon^{7/4} \phi^{(2)} + \dots \end{aligned} \quad (30)$$

Then in the lowest order in the expansion of equation (7) in powers of  $\varepsilon^{1/2}$  we get again equation (14) where  $v$  is replaced by  $2\rho^2/\sqrt{3}$ ,

$$\phi_{\xi}^{(1)} = \frac{2\sqrt{3}}{\rho} a^{(1)}. \quad (31)$$

In the next order we obtain the relationship

$$\phi_{\xi}^{(2)} = \frac{2\sqrt{3}}{\rho} a^{(2)} - \frac{1}{2\sqrt{3}\rho} a_{\xi\xi}^{(1)} \quad (32)$$

and finally in the highest relevant order we arrive at the fifth-order KdV equation

$$a_{\tau}^{(1)} + \frac{2\sqrt{3}}{\rho} \left(1 - \frac{4\rho^2}{3}\right) a^{(1)} a_{\xi}^{(1)} + \frac{\sqrt{3}\rho^2}{270} a_{\xi\xi\xi\xi\xi}^{(1)} = 0. \quad (33)$$

Note that here the nonlinear term can be obtained from the corresponding term in the KdV equation (4) by the substitution of  $\eta = \frac{1}{\rho^2} - \frac{1}{3}$  and the dispersion term reproduces the expansion of the dispersion relation (9) at the same value of  $\eta$ .

2.4. *KdV(2) equation*

The most interesting point corresponds to the values

$$\eta = 1 \quad \rho = \sqrt{3}/2 \tag{34}$$

when both nonlinear and dispersion coefficients vanish. Since equation (1) with  $\eta = 1$  coincides with the completely integrable AL equation, in a small amplitude approximation, it reduces again to a completely integrable equation. The KdV equation (4) with  $\eta = 1$  is meaningful for the entire interval  $0 < \rho < 1$  except for some vicinity of the point  $\rho = \sqrt{3}/2$ . Therefore one can suppose that at the point (34) one has to obtain the second equation of the KdV hierarchy—KdV(2), in which the higher order nonlinear and dispersion effects play the dominant role. In other words, now the cubic nonlinearity must be of the same order of magnitude as  $a^{(V)}$ ,  $a \sim \phi_x, a_t \sim a^2 a_x \sim a^{(V)}$ , which yields  $\{\alpha = \frac{5}{2}, \beta = \gamma = \frac{1}{2}\}$ , and the scaled variables

$$\tau = \varepsilon^{5/2} t \quad \xi = \varepsilon^{1/2} (x + vt) \quad v = \pm \frac{\sqrt{3}}{2} \tag{35}$$

where  $v$  is the velocity of linear waves at the point (34). The variables  $a(x, t)$  and  $\phi(x, t)$  have the same form of expansions (13) as in the KdV equation case. In the lowest order we obtain equation (23), in the next order we get the relationship

$$\phi_\xi^{(2)} = 4a^{(2)} + 4\sqrt{3}a^{(1)2} - \frac{1}{3}a_{\xi\xi}^{(2)} \tag{36}$$

and in the highest relevant order we obtain the KdV(2) equation

$$a_\tau^{(1)} + 16\sqrt{3}a^{(1)2}a_\xi^{(1)} - \frac{4}{3}a_\xi^{(1)}a_{\xi\xi}^{(1)} - \frac{2}{3}a^{(1)}a_{\xi\xi\xi}^{(1)} + \frac{\sqrt{3}}{360}a_{\xi\xi\xi\xi\xi}^{(1)} = 0. \tag{37}$$

As one should expect, the main nonlinear term here coincides with that of the mKdV equation (25) at the point (34), and the linear dispersion term with the corresponding term of the fifth-order KdV equation (33) at the same point.

By means of replacements

$$a^{(1)} = -\frac{1}{8\sqrt{3}}u \quad \tau = -30\sqrt{3}t \quad \xi = x$$

equation (37) can be transformed into the standard form of the second equation of the KdV hierarchy—the KdV(2) equation (see, e.g., [5]):

$$u_t = \frac{15}{2}u^2u_x + 5u_xu_{xx} + \frac{5}{2}uu_{xxx} + \frac{1}{4}u^{(V)}. \tag{38}$$

2.5. *Summary of the section*

Thus the GDNLS equation not even being integrable for  $\eta < 1$  is intimately related, through the small amplitude limit, to several standard integrable models: KdV, mKdV and KdV(2) equations, which appear for different choices of the deformation parameter  $\eta$  and the background amplitude. Another model appearing in the description of weak long-wavelength excitations of the GDNLS is the fifth-order KdV equation. Physically acceptable values of  $\eta$  and  $\rho$  are limited by the inequalities

$$0 \leq \eta \leq \min\{1, 1/\rho^2\} \quad 0 < \rho < \infty. \tag{39}$$

Figure 1 summarizes these results.

The completely integrable AL equation  $\eta = 1$  reduces in the small amplitude approximation either to the KdV equation, beyond some vicinity of the specific value of the background (34), or to the KdV(2) equation at the point (34), so that approximate equations remain completely integrable in both cases. This observation corroborates the fact that the



property of complete integrability is preserved in the framework of the singular perturbation scheme (see, e.g., [15]). This allows us to make a more general conjecture that higher equations of some hierarchy may arise as approximate equations if the underlying lattice depends on more than one parameter. This happens if at some values of the parameters of the underlying completely integrable problem nonlinearity and, hence, dispersion of lower equations of the hierarchy vanish. This phenomenon can be viewed as the physical meaning of the higher equations of hierarchies of integrable equations.

### 3. Dissipationless shock waves

In the dispersionless limit when dispersion effects can be neglected compared with nonlinear ones, all the above derived equations reduce in the leading approximation to the Hopf-like equation

$$u_t + u^n u_x = 0 \quad (40)$$

where  $n = 1$  for the KdV and fifth-order KdV equations and  $n = 2$  for mKdV and KdV(2) equations. It is well known (see, e.g., [5, 6]) that equation (40) with general enough initial condition leads to the formation of a wave-breaking point after which the solution becomes a multi-valued function of  $x$ . This means that near the wave-breaking point one cannot neglect the dispersion effects. If we take them into account, then the multi-valued region is replaced by the oscillatory region of the solution of the full equation. This oscillatory region is called the dissipationless shock wave and its analytical description is the aim of this section.

The existing theory of dissipationless shock waves is effective for completely integrable equations. Among equations derived in the preceding section, however, the fifth-order KdV equation (33) does not belong to this class. Fortunately, just this case of zero first-order dispersion was studied numerically in [10, 11]. We also bear in mind that the dissipationless shock waves of the KdV equation are already described in the literature [13] (see also [5]). Therefore we shall not consider this equation here and concentrate on the completely integrable models mKdV and KdV(2).

The analytical approach is based on the idea that the oscillatory region of the dissipationless shock wave can be represented as a modulated periodic solution of the equation under consideration. If the parameters defining the solution change little on a distance of one wavelength and during the time of the order of one period, one can distinguish two scales of time in this problem—fast oscillations of the wave and slow change of the parameters of the wave. Then equations which govern a slow evolution of the parameters can be averaged over fast oscillations, which leads to the so-called Whitham equations [1] and their solution subject to appropriate initial and boundary conditions describes the evolution of the dissipationless shock wave. This approach was suggested in [13] and now it is well developed for the KdV equation case (see, e.g., [5]). The results of this theory can be applied to shocks in the GDNLS equation when it is reduced to the KdV equation (4) or (19). We shall first develop an analogous theory for the mKdV and KdV(2) equations and then compare the results obtained for different equations.

#### 3.1. Dissipationless shock wave in the mKdV equation (26)

The mKdV equation corresponds to the line between bright KdV shock waves and dark KdV shock waves. And because the mKdV equation allows the transformation  $u = -u$ , on this line, we can expect both bright and dark shocks for the same values of the parameters  $\eta$  and  $\rho$ .

First we have to express a periodic solution of the mKdV equation (26) in a form suitable for the Whitham modulation theory. Such a form is provided automatically by the

finite-gap integration method which is used here to find the one-phase periodic solution of the mKdV equation.

The finite-gap integration method (see, e.g., [5]) is based on the complete integrability of the mKdV equation, following from a possibility of representing this equation as a compatibility condition  $\Psi_{xt} = \Psi_{tx}$  of two linear systems

$$\Psi_x = \mathbb{U}\Psi \quad \Psi_t = \mathbb{V}\Psi \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{41}$$

$$\mathbb{U} = \begin{pmatrix} -i\lambda & iu \\ -iu & i\lambda \end{pmatrix} \quad \mathbb{V} = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}$$

$$A = -4i\lambda^3 - 2iu^2\lambda \tag{42}$$

$$B = 4iu\lambda^2 - 2u_x\lambda - iu_{xx} + 2iu^3 \tag{43}$$

$$C = -4iu\lambda^2 - 2u_x\lambda + iu_{xx} - 2iu^3 \tag{44}$$

where  $\lambda$  is a free spectral parameter. The linear systems (41) have two basis solutions  $\Psi^\pm = (\psi_1^\pm, \psi_2^\pm)$ , from which we build the so-called squared basis functions,

$$f = -\frac{i}{2}(\psi_1^+\psi_2^- + \psi_1^-\psi_2^+) \quad g = \psi_1^+\psi_1^- \quad h = -\psi_2^+\psi_2^- \tag{45}$$

They satisfy the following linear systems,

$$f_x = -ug - uh \quad g_x = -2uf - 2i\lambda g \quad h_x = -2uf + 2i\lambda h \tag{46}$$

and

$$f_t = -iCg + iBh \quad g_t = 2iBf + 2Ag \quad h_t = -2iCf - 2Ah \tag{47}$$

and have the following integral,

$$f^2 - gh = P(\lambda) \tag{48}$$

independent of  $x$  and  $t$ . The periodic solutions are distinguished by the condition that  $P(\lambda)$  be a polynomial in  $\lambda$  with no zero coefficient (i.e.  $s_1, s_2, s_3 \neq 0$ ) and we shall see that the one-phase solution corresponds to the sixth degree polynomial in even powers of  $\lambda$ ,

$$P(\lambda) = \prod_{i=1}^3 (\lambda^2 - \lambda_i^2) = \lambda^6 - s_1\lambda^4 + s_2\lambda^2 - s_3. \tag{49}$$

Then  $f, g, h$ , satisfying (46)–(48), should also be polynomials in  $\lambda$ ,

$$f = \lambda^3 - f_1\lambda \quad g = iu(\lambda - \mu_1)(\lambda - \mu_2) \quad h = -iu(\lambda + \mu_1)(\lambda + \mu_2) \tag{50}$$

where  $\mu_j$  are new dependent variables. Substitution of (50) into (48) gives the conservation laws ( $s_j$  are constants)

$$2f_1 + u^2 = s_1 \quad f_1^2 + u^2(\mu_1^2 + \mu_2^2) = s_2 \quad u^2\mu_1^2\mu_2^2 = s_3 \tag{51}$$

and substitution into (46) and (47) yields the following important formulae:

$$u_x = 2iu(\mu_1 + \mu_2) \tag{52}$$

$$u_t = 2(2f_1 + u^2)u_x. \tag{53}$$

From (53) and the first equation (51) we see that  $u$  depends only on the phase

$$u = u(\theta) \quad \theta = x + 2s_1t \tag{54}$$

and from (52) and the other equations (51) we find

$$u_\theta^2 = u^4 - 2s_1 u^2 \mp 8\sqrt{s_3} u + s_1^2 - 4s_2 \equiv Q(u) \quad (55)$$

where the zeros  $u_i$  of the polynomial  $Q(u)$  are related to the zeros  $\lambda_i$  of the polynomial  $P(\lambda)$  by the formulae

$$u_1 = \pm(\lambda_1 + \lambda_2 + \lambda_3) \quad (56)$$

$$u_2 = \pm(\lambda_1 - \lambda_2 - \lambda_3) \quad (57)$$

$$u_3 = \pm(-\lambda_1 + \lambda_2 - \lambda_3) \quad (58)$$

$$u_4 = \pm(-\lambda_1 - \lambda_2 + \lambda_3). \quad (59)$$

If we order the zeros  $\lambda_i$  according to

$$\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 \quad (60)$$

then for the upper choice of the sign in (55) and (56) we have

$$u_1 > u_2 > u_3 > u_4 \quad (61)$$

and  $u$  oscillates within the interval

$$u_3 \leq u \leq u_2 \quad (62)$$

where  $Q(u) \geq 0$ . For the lower choice of the sign in (55) and (56) we have

$$u_1 < u_2 < u_3 < u_4 \quad (63)$$

and  $u$  oscillates within the interval

$$u_2 \leq u \leq u_3. \quad (64)$$

We are interested in wave trains against a positive constant background which corresponds to the lower choice of sign in (55) and (56). In this case equation (55) yields the periodic solution

$$u(\theta) = \frac{(u_3 - u_1)u_2 - (u_3 - u_2)u_1 \operatorname{sn}^2(\sqrt{(u_4 - u_2)(u_3 - u_1)}\theta/2, m)}{u_3 - u_1 - (u_3 - u_2) \operatorname{sn}^2(\sqrt{(u_4 - u_2)(u_3 - u_1)}\theta/2, m)} \quad (65)$$

where

$$m = \frac{(u_3 - u_2)(u_4 - u_1)}{(u_4 - u_2)(u_3 - u_1)} = \frac{\lambda_1^2 - \lambda_2^2}{\lambda_1^2 - \lambda_3^2} \quad \theta = x + 2s_1 t = x + 2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)t. \quad (66)$$

At  $\lambda_2 = \lambda_3$ , when  $m = 1$ , the solution (65) transforms into a soliton solution of the mKdV equation

$$u_s(\theta) = \lambda_1 - \frac{2(\lambda_1^2 - \lambda_2^2)}{\lambda_1 - \lambda_2 + 2\lambda_2 \cosh^2(2\sqrt{\lambda_1^2 - \lambda_2^2}\theta)} \quad \theta = x + 2(\lambda_1^2 + 2\lambda_2^2)t. \quad (67)$$

In a modulated wave the parameters  $\lambda_i$  become slow functions of  $x$  and  $t$ . It is convenient to introduce new variables

$$r_1 = \lambda_3^2 \quad r_2 = \lambda_2^2 \quad r_3 = \lambda_1^2 \quad (68)$$

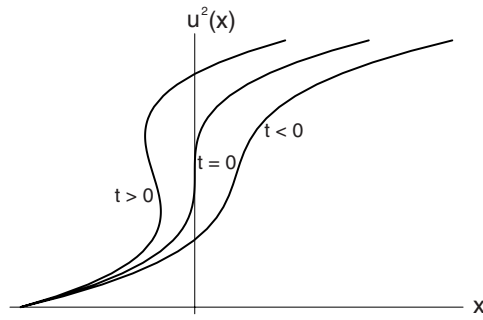
so that the Whitham equations can be written in the form (see, e.g., [5])

$$\frac{\partial r_i}{\partial t} + v_i(r) \frac{\partial r_i}{\partial x} = 0 \quad v_i = \left(1 - \frac{L}{\partial_i L} \partial_i\right) V \quad i = 1, 2, 3 \quad (69)$$

where  $V = -2(r_1 + r_2 + r_3)$  is the phase velocity of the nonlinear wave (65) and

$$L = \frac{K(m)}{\sqrt{r_3 - r_1}} \quad m = \frac{r_3 - r_2}{r_3 - r_1} \quad (70)$$

is the wavelength.



**Figure 2.** Formation of the multi-valued solution of the mKdV equation in the dispersionless limit (71). The initial data correspond to the cubic curve  $x = -(u^2 - u_b^2)^3$  (see equation (73)).

Now our task is to consider the solution of the mKdV equation after the wave-breaking point. As follows from (25), before this point in the dispersionless approximation the evolution of the pulse obeys the Hopf equation

$$u_t - 6u^2u_x = 0 \tag{71}$$

with the well-known solution

$$x + 6u^2t = f(u^2) \tag{72}$$

where  $f(u^2)$  is determined by the initial condition. At the wave-breaking point, which will be assumed to be  $t = 0$ , the profile  $u^2(x)$  has an inflection point with vertical tangent line,

$$\left. \frac{\partial x}{\partial u^2} \right|_{t=0} = 0 \quad \left. \frac{\partial^2 x}{\partial (u^2)^2} \right|_{t=0} = 0.$$

Hence, in its vicinity we can represent (72) as

$$x + 6u^2t = -(u^2 - u_b^2)^3 \tag{73}$$

where  $u_b = u(x_b, t_b)$ . Note that the mKdV equation is not Galileo invariant and therefore we cannot eliminate the constant parameter  $u_b^2$ , in contrast to the case of a KdV equation (see, e.g., [5]).

For  $t > 0$  solution (73) becomes a multi-valued function of  $x$ . Formation of this multi-valued region is shown in figure 2. For  $t \geq 0$  we cannot neglect dispersion and have to consider the full mKdV equation. Due to the effect of dispersion, the multi-valued region is replaced by the region of fast oscillations which can be represented as a modulated periodic solution of the mKdV equation (26). We rewrite this solution (see equations (65) and (66)) in terms of the slowly varying functions  $r_i(x, t)$ ,  $i = 1, 2, 3$ ,

$$u(x, t) = \frac{(\sqrt{r_3} + \sqrt{r_1})(\sqrt{r_2} + \sqrt{r_1} - \sqrt{r_3})}{\sqrt{r_1} + \sqrt{r_3} - (\sqrt{r_3} - \sqrt{r_2}) \operatorname{sn}^2(2\sqrt{r_3} - r_1\theta, m)} + \frac{(\sqrt{r_3} - \sqrt{r_2})(\sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3}) \operatorname{sn}^2(2\sqrt{r_3} - r_1\theta, m)}{\sqrt{r_1} + \sqrt{r_3} - (\sqrt{r_3} - \sqrt{r_2}) \operatorname{sn}^2(2\sqrt{r_3} - r_1\theta, m)} \tag{74}$$

where

$$m = \frac{r_3 - r_2}{r_3 - r_1} \quad \theta = x + 2s_1t = x + 2(r_1 + r_2 + r_3)t. \tag{75}$$

and functions  $r_i(x, t)$  are governed by the Whitham equations (69). We have to find such a solution of these equations that the region of oscillations matches at its end points

corresponding to  $m = 0$  and  $m = 1$  the dispersionless solution (73), which we rewrite in the form

$$x + 6rt = -(r - r_b)^3 \quad r = u^2 \quad r_b = u_b^2. \quad (76)$$

This means that the solution of equations (69) written in implicit form

$$x - v_i(r)t = w_i(r) \quad i = 1, 2, 3 \quad (77)$$

must satisfy the boundary conditions

$$v_1|_{r_2=r_3} = -6r_1 \quad v_3|_{r_2=r_1} = -6r_3 \quad (78)$$

$$w_1|_{r_2=r_3} = -(r_1 - r_b)^3 \quad w_3|_{r_2=r_1} = -(r_1 - r_b)^3. \quad (79)$$

Then, as we shall see from the results, the mean values of  $u$  will match at these boundaries the solution of the dispersionless mKdV equation.

To find the solution (77) subject to the boundary conditions (78),(79), we shall follow the method developed earlier for the KdV equation (see, e.g., [5]). We look for  $w_i$  in a form similar to (69),

$$w_i = \left(1 - \frac{L}{\partial_i L} \partial_i\right) W \quad i = 1, 2, 3 \quad (80)$$

and find that  $W$  satisfies the Euler–Poisson equation

$$\partial_i W - \frac{1}{2(r_i - r_j)} (\partial_i W - \partial_j W) = 0 \quad i \neq j. \quad (81)$$

For our aim it is enough to know a particular solution of this linear equation  $W = \text{const}/\sqrt{P(r)}$ , where  $P(r)$  is a polynomial with zeros  $r_i$  and it can be identified with polynomial (49) taking into account equations (68). The series expansion of this solution in inverse powers of  $r$ ,

$$W = \frac{-4r^{3/2}}{\sqrt{(r - r_1)(r - r_2)(r - r_3)}} = \sum_{n=0}^{\infty} \frac{W^{(n)}}{r^n} \quad (82)$$

can be considered as the generating function of a sequence of solutions

$$W^{(1)} = -2s_1 \quad W^{(2)} = 2s_2 - \frac{3}{2}s_1^2 \quad W^{(3)} = 3s_1s_2 - 2s_3 - \frac{5}{4}s_1^3 \quad (83)$$

where  $s_1, s_2, s_3$  are the coefficients of the polynomial (49) expressed in terms of  $r_1, r_2, r_3$ :

$$s_1 = r_1 + r_2 + r_3 \quad s_2 = r_1r_2 + r_1r_3 + r_2r_3 \quad s_3 = r_1r_2r_3. \quad (84)$$

It is easy to find that the resulting velocities

$$w_i^{(n)} = \left(1 - \frac{L}{\partial_i L} \partial_i\right) W^{(n)} \quad i = 1, 2, 3 \quad (85)$$

have the following limiting values

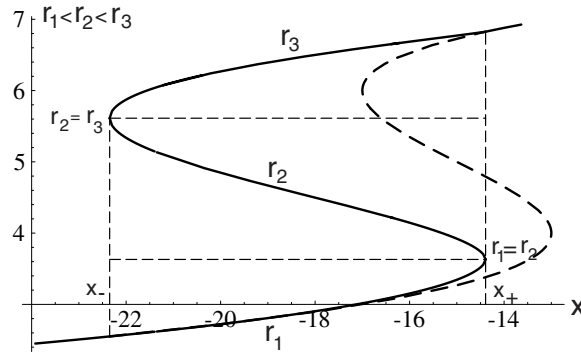
$$w_1^{(1)} \Big|_{r_2=r_3} \equiv v_1|_{r_2=r_3} - 6r_1 \quad w_3^{(1)} \Big|_{r_2=r_1} \equiv v_3|_{r_2=r_1} = -6r_3 \quad (86)$$

$$w_1^{(2)} \Big|_{r_2=r_3} = -\frac{15}{2}r_1^2 \quad w_3^{(2)} \Big|_{r_2=r_1} = -\frac{15}{2}r_3^2 \quad (87)$$

$$w_1^{(3)} \Big|_{r_2=r_3} = -\frac{35}{4}r_1^3 \quad w_3^{(3)} \Big|_{r_2=r_1} = -\frac{35}{4}r_3^3. \quad (88)$$

Thus, we see that if we take

$$w_i(r) = -r_b^3 - \frac{1}{2}r_b^2 w_i^{(1)}(r) + \frac{2}{5}r_b w_i^{(2)}(r) - \frac{4}{35}w_i^{(3)}(r) \quad i = 1, 2, 3 \quad (89)$$



**Figure 3.** Dependence of the Riemann invariants  $r_1, r_2, r_3$  on  $x$  at some fixed value of time. The plots are calculated according to formulae (77) with  $r_b = 5$  and  $t = 0.5$ . A dashed line represents the dispersionless solution which matches the Riemann invariants at the boundaries  $x_{\pm}$  of the region of oscillations.

then formulae (77) satisfy all necessary conditions and define the dependence of  $r_1, r_2, r_3$  on  $x$  and  $t$  in implicit form. In figure 3 we show the dependence of  $r_1, r_2, r_3$  on  $x$  at  $t = 0.5$  and  $r_b = 5$ . It is clearly seen that  $r_2$  and  $r_1$  coalesce at the right boundary  $x_+$ , where  $m = 1$ , and  $r_2$  and  $r_3$  coalesce at the left boundary  $x_-$ , where  $m = 0$ . The dispersionless solution is depicted by a dashed line and  $r_1$  matches this solution at  $x_-$  and  $r_3$  matches it at  $x_+$ .

Let us find the laws of motion  $x_{\pm}(t)$  of the boundaries of the region of oscillations. At the right boundary we have the condition

$$\left. \frac{dx}{dr_1} \right|_{m=1} = \left. \frac{dx}{dr_2} \right|_{m=1} = 0 \tag{90}$$

which yields the expression

$$t = \frac{12}{35}r_1^2 + \frac{4}{35}r_1r_3 + \frac{3}{70}r_3^2 - \frac{4}{5}r_1r_b - \frac{1}{5}r_3r_b + \frac{1}{2}r_b^2 \tag{91}$$

and substitution of this expression into equations (77) with  $r_1 = r_2$  gives the coordinate  $x_+$  expressed in terms of the Riemann invariants  $r_1$  and  $r_3$ :

$$x_+ = \frac{1}{35}[-32r_1^3 + 2r_3^3 - 8r_1r_3(r_3 - 7r_b) - 8r_1^2(4r_3 - 7r_b) - 7r_3^2r_b - 35r_b^3]. \tag{92}$$

On the other hand, this value of  $x_+$  must coincide with the coordinate obtained from the dispersionless solution (76) with  $t$  equal to equation (91),

$$\begin{aligned} x_+ &= -6r_3t - (r_3 - r_b)^3 \\ &= \frac{1}{35}[-72r_1^2r_3 + 26r_3^3 - 24r_1r_3(r_3 - 7r_b) - 63r_3^2r_b - 35r_b^3]. \end{aligned} \tag{93}$$

Comparison of these two expressions for  $x_+$  yields the relation between  $r_1, r_2$  and  $r_3$  at  $m = 1$ :

$$(4r_1 + 3r_3)_{m=1} = 7r_b. \tag{94}$$

Substitution of  $r_1$  obtained from this equation into equation (91) gives

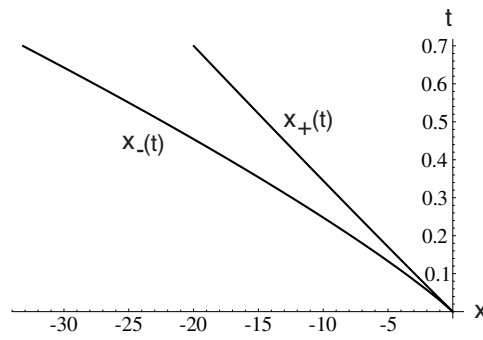
$$t = \frac{3}{20}(r_3 - r_b)^2. \tag{95}$$

Hence

$$r_3|_{m=1} = r_b + \frac{2}{3}\sqrt{15t} \tag{96}$$

and again with the use of equation (94) we obtain

$$r_1|_{m=1} = r_2|_{m=1} = r_b - \frac{1}{2}\sqrt{15t}. \tag{97}$$



**Figure 4.** Dependence of coordinates  $x_{\pm}$  of the boundaries of the region of oscillations on time for the mKdV equation case.

These formulae give values of the Riemann invariants at the right boundary as functions of time  $t$ . Their substitution into (92) or (93) yields the motion law of the right boundary

$$x_+(t) = -6r_b t + \frac{4}{3} \sqrt{\frac{5}{3}} t^{3/2}. \quad (98)$$

In a similar way at the left boundary  $x_-$  the conditions

$$\left. \frac{dx}{dr_2} \right|_{m=0} = \left. \frac{dx}{dr_3} \right|_{m=0} = 0 \quad (99)$$

yield

$$t = -\frac{1}{30} r_1^2 - \frac{4}{15} r_1 r_3 + \frac{4}{5} r_3^2 + \frac{1}{3} r_1 r_b - \frac{4}{3} r_3 r_b + \frac{1}{2} r_b^2 \quad (100)$$

which substitution into equations (77) and (76) gives, respectively,

$$x_- = \frac{1}{5} [-2r_1^3 - 32r_3^3 + 8r_1 r_3 (4r_3 - 5r_b) + 40r_3^2 r_b - r_b^3 + r_1^2 (-8r_3 + 15r_b)] \quad (101)$$

and

$$x_- = \frac{1}{5} [6r_1^3 + r_1^2 (8r_3 - 25r_b) - 8r_1 r_3 (3r_3 - 5r_b) - 5r_b^3]. \quad (102)$$

Their comparison yields the relation

$$(r_1 + 4r_3)_{m=0} = 5r_b \quad (103)$$

which permits us to eliminate  $r_3$  from equation (100) to obtain

$$t = \frac{1}{20} (r_1 - r_b)^2. \quad (104)$$

Hence

$$r_1|_{m=0} = r_b - 2\sqrt{3t} \quad (105)$$

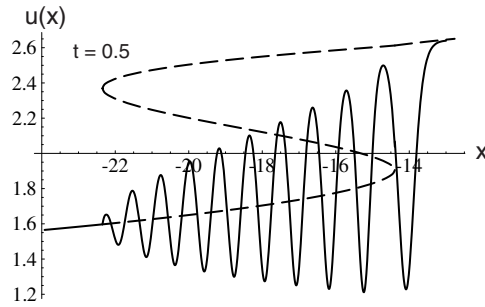
and again with the use of equation (94) we obtain

$$r_2|_{m=0} = r_3|_{m=0} = r_b + \frac{1}{2}\sqrt{3t}. \quad (106)$$

Substitution of these formulae into (101) or (102) yields the motion law of the left boundary

$$x_-(t) = -6r_b t - 12\sqrt{3} t^{3/2}. \quad (107)$$

The plots of  $x_+(t)$  and  $x_-(t)$  are depicted in figure 4. To the right from  $x_+(t)$  and to the left from  $x_-(t)$  the wave is described by the dispersionless solution (76). Between  $x_+(t)$  and  $x_-(t)$  we have the region of fast oscillations represented by equation (75) with  $r_i(x, t)$ ,  $i = 1, 2, 3$ , given implicitly by equations (89). The dependence of  $u$  on  $x$  at some fixed moment of time is shown in figure 5. It describes a dissipationless shock wave connecting two smooth regions where we can neglect dispersion effects. At the right boundary, the periodic wave tends to a sequence of separate dark soliton solutions of the mKdV equation and at the left boundary the amplitude of oscillations tends to zero.



**Figure 5.** The dissipationless shock wave for the mKdV equation. The parameters are equal to  $r_b = 5$  and  $t = 0.5$ . The dashed line represents the square roots of the Riemann invariants which match the smooth solution  $u(x, t)$  of the dispersionless equation at the boundaries of the region of oscillations.

3.2. Dissipationless shock wave in the KdV(2) equation (38)

The theory of dissipationless shock wave for the KdV(2) equation is similar to that for the KdV and mKdV cases. Therefore we shall present here only its main points.

The periodic solution of the KdV(2) equation has the same form as in the KdV equation case (see, e.g., [5]),

$$u(x, t) = r_2 + r_3 - r_1 - 2(r_2 - r_1) \operatorname{sn}^2(\sqrt{r_3 - r_1}\theta, m) \quad m = \frac{r_2 - r_1}{r_3 - r_1} \quad (108)$$

with the phase velocity

$$\theta = x - Vt \quad V = 2s_2 - \frac{3}{2}s_1^2 \quad (109)$$

corresponding to the second equation of the KdV hierarchy.

Now the periodic solution (108) is parametrized by the Riemann invariants  $r_i, i = 1, 2, 3$ , rather than by their squared roots, as it was in the mKdV equation case. Hence, the solution of the dispersionless equation

$$u_t = \frac{15}{2}u^2u_x \quad (110)$$

near the wave-breaking point should be taken in the form

$$x + \frac{15}{2}u^2 = -(u - u_b)^3. \quad (111)$$

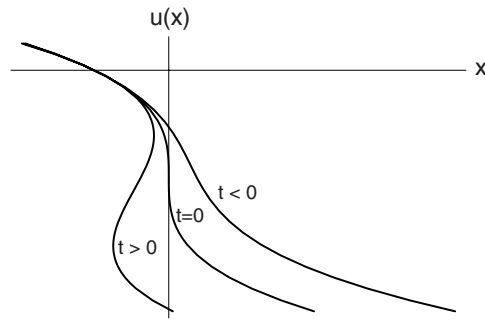
In fact, this form is equivalent near the wave-breaking point to the solution (73), since  $(u^2 - u_b^2) \simeq 2u_b(u - u_b)$  and the constant factor can be scaled out. Formation of multi-valued region is illustrated in figure 6. After taking into account the dispersion effects it should be replaced by the dissipationless shock wave.

Within the shock wave we have modulated periodic solution (108) where  $r_i$  are slow functions of  $x$  and  $t$  and their evolution is governed by the Whitham equations (69) with  $V$  defined by equation (109). Their solution subject to the necessary boundary conditions can be found by the same method as was used in the preceding subsection. As a result we obtain

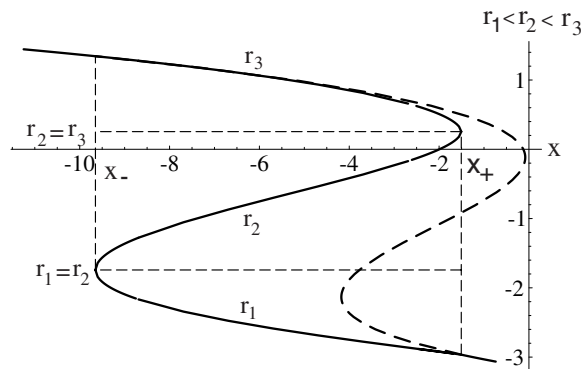
$$x - w_i^{(2)}t = u_b^3 + \frac{1}{2}u_b^2w_i^{(1)} - \frac{2}{5}u_bw_i^{(2)} + \frac{4}{35}w_i^{(3)} \quad (112)$$

where  $w_i^{(n)}$  are defined by formulae (83)–(85). Equations (112) define implicitly the dependence of the Riemann invariants  $r_i, i = 1, 2, 3$ , on  $x$  and  $t$ . The resulting plots are shown in figure 7. At the right boundary  $x_+$  we have a soliton limit ( $m = 1$ ) and at the left boundary  $x_-$  we have a wave with vanishing modulation.

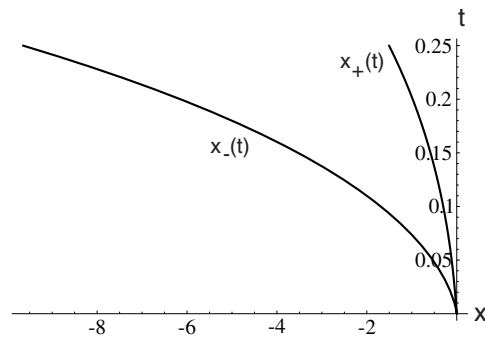




**Figure 6.** Formation of the multi-valued solution of the KdV(2) equation in the dispersionless limit (110). The initial data correspond to the cubic curve  $x = -(u - u_b)^3$  (see equation (111)).

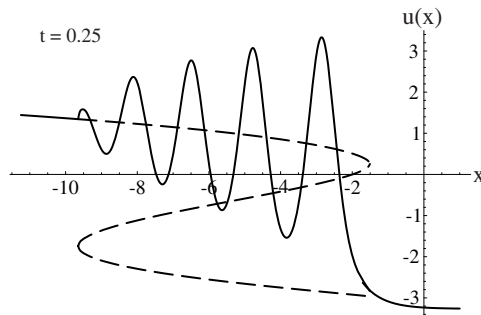


**Figure 7.** Dependence of the Riemann invariants  $r_1, r_2, r_3$  on  $x$  at some fixed value of time for the KdV(2) equation case. The plots are calculated according to formulae (112) with  $u_b = -0.5$  and  $t = 0.25$ . A dashed line represents the dispersionless solution which matches the Riemann invariants at the boundaries  $x_{\pm}$  of the region of oscillations.



**Figure 8.** Dependence of coordinates  $x_{\pm}$  of the boundaries of the region of oscillations on time for the KdV(2) equation case.

The motion laws  $x_{\pm}(t)$  can be found as in the preceding subsection, but the final formulae now become quite complicated and we shall not write them down. The corresponding plots are presented in figure 8. Again the region between  $x_-(t)$  and  $x_+(t)$  corresponds to an expanding



**Figure 9.** The dissipationless shock wave for the KdV(2) equation. The parameters are equal to  $u_b = -0.5$  and  $t = 0.25$ . The dashed line represents the Riemann invariants which match the smooth solution  $u(x, t)$  of the dispersionless equation at the boundaries of the region of oscillations.

with time  $t$  dissipationless shock wave. It is illustrated in figure 9 where the dependence  $u(x)$  at the fixed moment  $t$  is shown. Now at one boundary bright solitons are formed and at the other boundary the amplitude of oscillations tends to zero.

#### 4. Numerical analysis

The theory developed in the previous section describes the behaviour of smooth and weak enough perturbations against the background. Such pulses can evolve towards the creation of shock waves giving rise to oscillatory fronts, which can be interpreted as modulated periodic waves [13]. In order to complete the theory developed, in the present section we provide direct numerical simulations of the GDNLS equation illustrating creation and evolution of shock waves. More specifically, we consider different kinds of shock waves occurring at different values of the background amplitude  $\rho$ , and the constant deformation parameter  $\eta = 0.15$ .

The chain is taken to be long enough,  $N = 2100$ , to ignore effects due to boundary conditions which are taken as  $q_0(t) = q_1(t)$ ,  $q_{N+1}(t) = q_N(t)$ . We take a zero derivative for boundary points to impose that we are far enough to have a constant value of the function at the boundary points. The initial condition in all simulations except the mKdV case was taken as

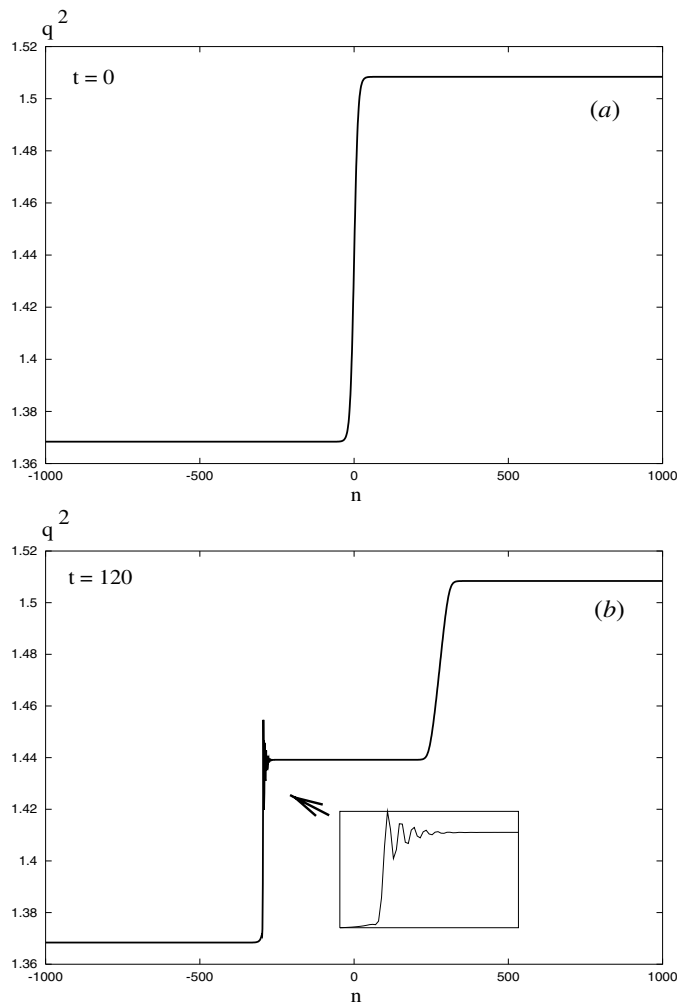
$$q_n(0) = \rho + 0.07 \tanh(0.08n). \tag{113}$$

We perform a numerical analysis for values of the parameter  $\rho$  corresponding to different regions of figure 1.

##### 4.1. The first KdV region

For  $\rho = 1$ , the GDNLS equation can be reduced to the KdV equation (4) and the evolution of the initial condition (113) governed by the GDNLS equation is shown in figure 10.

At the initial stage of the evolution two step-like pulses are formed—one propagating to the right and the other propagating to the left. We note that this is a typical situation for the decay of a step-like pulse within the framework of the KdV equation (see, e.g., [13]). The wave propagating to the right is a rarefaction wave and in long-time evolution it does not decay in a train of solitons. In contrast, the left propagating wave breaks down due to nonlinear effects at the moment about  $t_b \simeq 72$ . After that time, oscillations appear at the wave front which is clearly seen in the inset of figure 10(b).

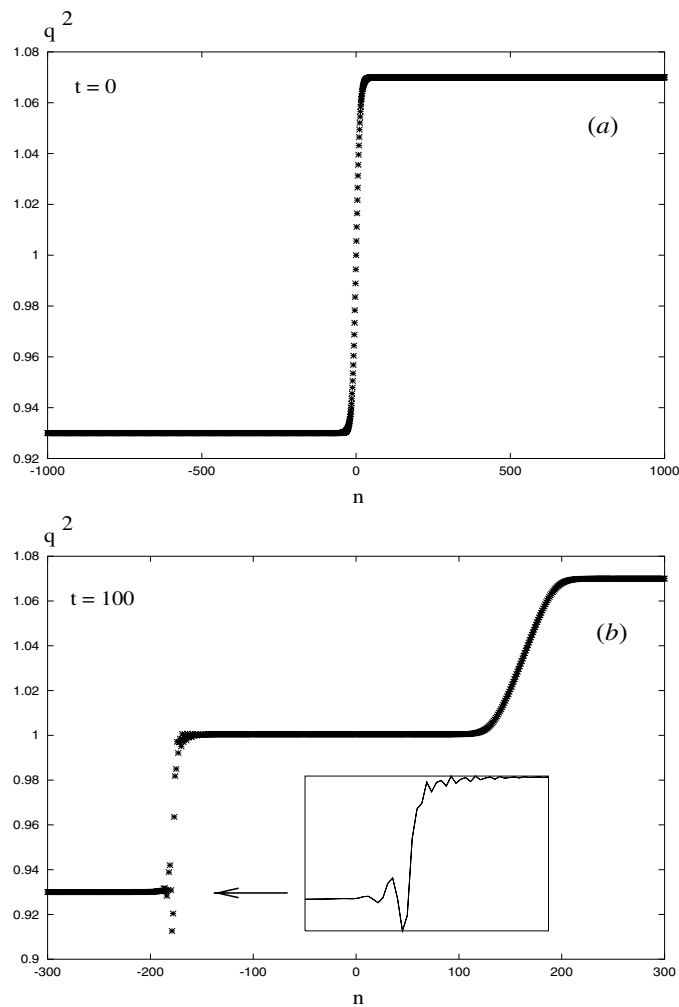


**Figure 10.** Evolution of the initial front perturbation with  $\rho = 1.44$  at  $t = 0$  (a) and  $t = 120$  (b). The inset in (b) shows details of the wave front after a shock develops.

This breaking time is principally due to the time needed for the initial front to evolve to a step front. This time is proportional to the difference in the velocity of the extreme points of the initial front  $\Delta v = v(\rho = 1) - v(\rho = 0.93) = 0.108$ . (The initial front is composed of at least 50 discrete sites in figure 10(a) and evolves to a step front with less than 10 sites in figure 10(b).) The shock appears at  $\Delta v \cdot t_b \simeq 7$ , which means that the shock is created when there are less than eight particles in the front.

#### 4.2. Fifth-order KdV limit

In this case  $\rho = 1.44$  corresponds to the line (17), and the dispersive terms differ from the previous case and lead to weaker dispersion effects. The continuous limit corresponds now to the fifth-order KdV equation (33).



**Figure 11.** Evolution of the initial front perturbation with  $\rho = 1$  at  $t = 0$  (a) and  $t = 100$  (b). The inset in (b) shows details of the wave front after a shock develops.

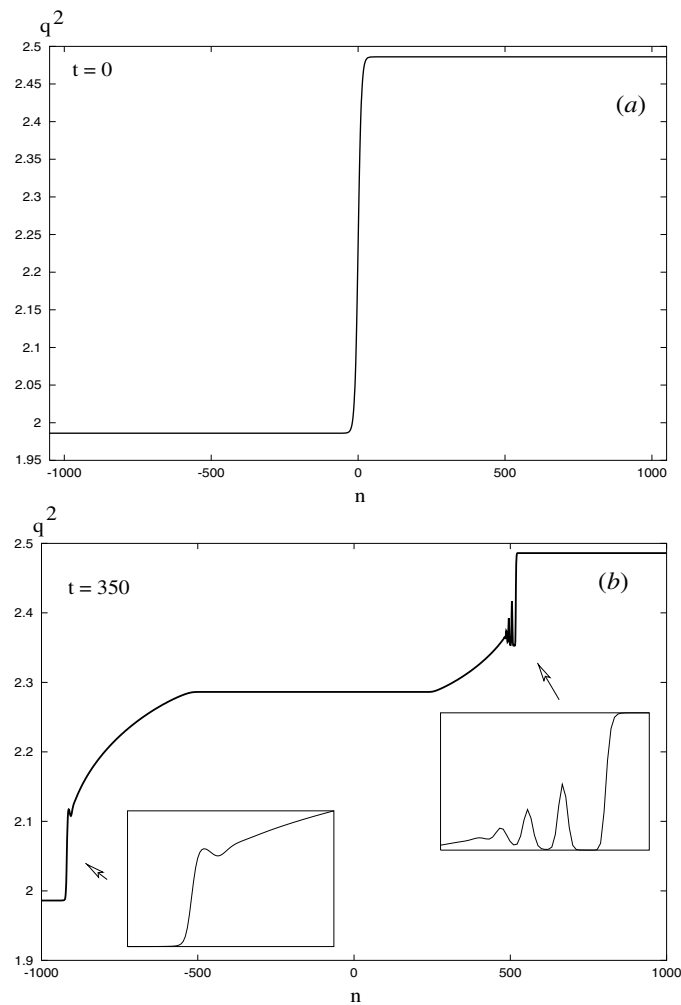
Formation of a shock in figure 11 occurs after the breaking time  $t_b \simeq 95$ . Here the difference of velocities between the extremum points of the initial front is equal to  $\Delta v = 0.068$ , that is it is less than in the KdV case. Hence the breaking time is smaller.

#### 4.3. The second KdV region

When we take  $\rho = 1.5$  and  $\eta = 0.15$ , the breaking time is equal to  $t_b \simeq 93$  in agreement with the difference in the velocities  $\Delta v = 0.061$ . Now the dispersion terms are negative and we obtain a shock in the upper part of the step-like pulse.

#### 4.4. mKdV region

On the mKdV line (16) we take  $\rho = 2.23$  for  $\eta = 0.15$ . The higher order of the nonlinearity term in the mKdV equation compared with the one in the KdV equation case means that for a

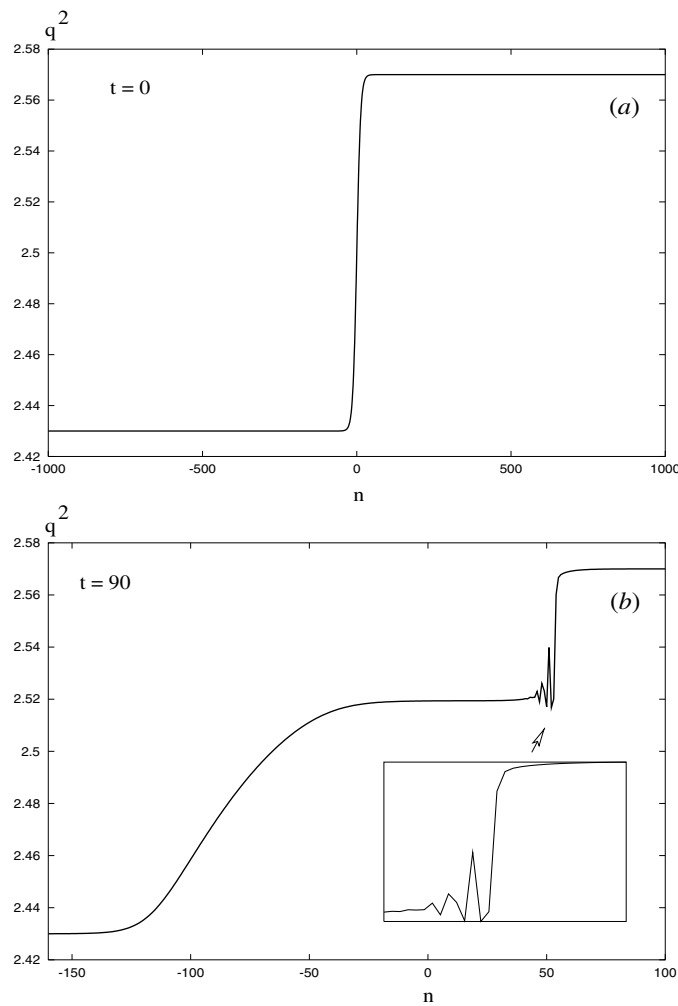


**Figure 12.** Evolution of initial front perturbation for  $\rho = 2.23$  at time  $t = 0$  (a) and  $t = 350$  (b). The insets in (b) show details of the wave front after a shock develops.

small amplitude of modulation the nonlinearity effects are weaker and the wave evolves to the wave-breaking point followed by the formation of a shock at a later moment of time (see (20)). To obtain the shock formation numerically, we need a sharper and stronger initial perturbation than the one given by (113). Thus now we take

$$q_n(0) = \rho + 0.25 \tanh(0.15n). \quad (114)$$

Figure 12 shows the formation of a shock after a time about  $t_b \simeq 150$  for the right part and  $t_b = 300$  for the left part of the pulse. Since the mKdV equation supports bright and dark solitons, both types can be generated during evolution of the wave. In figure 12(b) shocks appear first on the right side as bright solitons and on the left side start to appear also as dark solitons.



**Figure 13.** Evolution of initial step perturbation for  $\rho = 2.5$  at initial time (a) and  $t = 90$  (b). The inset in (b) shows details of the wave front after a shock develops.

#### 4.5. The third KdV region

For a value of the parameter  $\rho = 2.5$ , the nonlinearity term in equation (4) changes its sign compared with the previous cases  $\rho = 1$  and  $\rho = 1.5$ , but the sign of the velocity in front of the GDNLS equation (1) is still the same. Correspondingly, solitons move to the right and shock wave is formed in the wave propagating to the right. This prediction is confirmed by the numerical results shown in figure 13. The time-breaking point is  $t_b = 48$  in agreement with the difference of velocities  $\Delta v = 0.596$ .

## 5. Conclusion

In the present paper we have shown that the GDNLS equation with finite density boundary conditions can be reduced, depending on the values of the parameters  $\eta$  and  $\rho$ , to several

important continuous models—KdV, mKdV, KdV(2) and fifth-order KdV equations which describe different regimes of wave propagation in the nonlinear lattice.

The evolution equations obtained by means of the multiple-scale expansion as approximations to the GDNLS equation lead in the dispersionless limit for general enough initial pulses to wave-breaking and small dispersion effects yield formation of dissipationless shock waves. The shock waves display very different behaviour in different regions of the parameters defined by the small amplitude limit. We presented numerical observations of different types of shocks created from an initial step-like pulse. The theory of these waves is developed in the framework of the Whitham averaging method. Analytical expressions which describe the main characteristics of waves—trailing and leading end points, amplitudes and wavelengths—are obtained.

The phenomena described in the present paper are not restricted to the GDNLS equation, but are characteristic features of a large class of nonlinear Schrödinger lattices, which depend on one or more free parameters.

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